

# New strategy for Fractional Order System Modelling by using Multi-model Approach: a Thermal Application

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**Abstract**— In this study, a new strategy for modelling a fractional order system by multi-model approach is proposed and developed. It consists in using Self-Organization Map, named also KOHONEN Maps, to determine the local models. Identifying parameters of each local model using recursive least square algorithm and obtaining the multi-model output using fusion of local outputs are the two next steps. This method is applied to a fractional model of thermal system and it prove a good performance to imitate the behaviour of real system and allow to facilitate the study of such system .

**Keywords**— Fractional order system, thermal system, multi-model approach, Self-organization Map, Fractional model

## I. INTRODUCTION

The fractional order systems are systems that take into consideration the hereditary effects. Due to this characterization, many physical phenomena are modeled by a fractional transfer function such as electro elasticity, velocity, and diffusion [5].

The corresponding transfer function is an irrational function. Thus, an analytical solution for simulation and computation of fractional model's output is often difficult to obtain [8], [21]. To simplify it, many researchers use mathematical approximations of irrational function as Continued fraction expansion CFE, Power Series Expansion PSE, recursive approximation of Oustaloup... [22] to approximate the irrational transfer function on a finite-dimension transfer function. The obtained function is generally a high order function.

In order to facilitate the study of the obtained high order transfer function, we propose to represent fractional order model by multi-model approach with integer order local models. So, we propose the use of Self-Organizing Map to classify measurements and the Recursive Least Square to identify the local models. The multi-model output is obtained by fusion of local outputs weighted by their respective validities.

The effectiveness of the proposed method is proved by applying it to the thermal system model identified in [10].

This paper is organized as follows: the next section introduces the fractional calculus and presents the Recursive approximation of Oustaloup and discretization methods of fractional systems. Section 3 describes the multi-model approach. In section 4, the proposed method of modeling fractional order system by multi-model approach is represented. Section 5 emphasizes the application of the proposed method to a fractional function of thermal system. Results are discussed also in this section. Finally, conclusions and perspectives work are given.

## II. FRACTIONAL ORDER SYSTEMS

### A. Overview

From a physical point of view, to improve the precision of modeling of some phenomena, memory and hereditary effects need to be considered in the mathematical model. Using Fractional derivation and integration present a remedy.

Each fractional order system can be represented by the following fractional order differential equation:

$$a_n D^{\alpha_n} y(t) + a_{n-1} D^{\alpha_{n-1}} y(t) + \dots + a_1 D^{\alpha_1} y(t) + a_0 y(t) = b_m D^{\beta_m} u(t) + b_{m-1} D^{\beta_{m-1}} u(t) + \dots + b_1 D^{\beta_1} u(t) + b_0 u(t) \quad (1)$$

Where  $D^{\alpha_n}$ ,  $D^{\beta_{m-1}}$  are fractional derivative operators  $\alpha_n$ , and  $\beta_m$  are the fractional order.

The most known definitions of non-integers derivation and integration operators are [1], [2], [3]:

1) Grünwald-Letnikov definition [2]:

$$D^{\nu} f(t) = \frac{1}{h^{\nu}} \sum_{k=0}^{\infty} (-1)^k \binom{\nu}{k} f(t - kh) \quad (2)$$

Where  $t=Kh$ ,  $K \in \mathbb{N}$  and  $h$  is the sampling period.

2) Riemann-Liouville definition [3]:

$$D_t^\nu f(t) = \frac{d^n}{dt^n} \left[ \frac{1}{\Gamma(1-\nu)} \int_c^t \frac{f(\tau)}{(t-\tau)^{\nu-n+1}} d\tau \right] \quad (3)$$

Where  $n-1 < \nu < n$ ,  $n \in \mathbb{N}$ ,  $\Gamma(\nu) = \int_0^\infty x^{\nu-1} e^{-x} dx$  is the gamma function.

3) Caputo definition [1]:

$$D_t^\nu f(t) = \frac{1}{\Gamma(1-\nu)} \int_0^t \frac{f^{(1)}(\tau)}{(t-\tau)^\nu} d\tau \quad (4)$$

Where  $0 < \nu < 1$ ,  $y^{(1)}(t)$  is the first order derivative and  $\Gamma$  is the Euler gamma function.

To obtain the fractional transfer function, the application of Laplace transformation is required.

Assuming that  $f(t) = 0, \forall t < 0$ , the Laplace transform is:  $\mathcal{L}(D^\nu f(t)) = s^\nu \mathcal{L}(f(t))$  (5)

The term  $s^\alpha$  is irrational and infinite dimensional. It must receive a rational approximation to obtain an integer order transfer function in order to use z transformation for discretization.

B. Recursive approximation of Oustaloup

Many researchers describe rational approximations using PSE (power series expansion), CFE (Continued Fraction expansion), Carlson's approximation, Charef's approximation and Oustaloup's recursive approximation. To see more details about these approximations, you can see [5] and [6]. In [6] also, authors present a comparative study between these approximation methods.

In this paper, we are going to use Oustaloup's recursive approximation. It is given by the following equation [7], [4]:

$$s^\alpha \approx K \prod_{k=1}^N \frac{s+w'_k}{s+w_k} \quad (6)$$

Where,

$$w'_k = w_l w_u^{(2k-1-\alpha)/N} \quad (7)$$

$$w_k = w_l w_u^{(2k-1+\alpha)/N} \quad (8)$$

$$K = w_h^\alpha \quad (9)$$

$$w_u = \sqrt{w_h/w_l} \quad (10)$$

Where  $w_h$  and  $w_l$  are higher and lower frequency values and  $N$  is the order of the finite transfer function.

C. Discretization

There are two types of discretization methods: direct method and indirect method.

1) Direct method

The first one substitutes the Laplace operator 's' by a generic function ( see Table1) then applies a rational approximation (generally CFE and PSE approximation ) to obtain the discrete transfer function. This method is used for discretization of the fractional order operator in order to PI<sup>λ</sup>D<sup>λ</sup> controllers.

TABLE I  
 GENERICS FUNCTIONS

s→z	Method
$s^\alpha \approx \left(\frac{1}{T} \frac{1-z^{-1}}{z^{-1}}\right)^\alpha$	Forward Euler rule
$s^\alpha \approx \left(\frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha$	Trapezoidal rule (TUSTIN)
$s^\alpha \approx \left(\frac{8}{7T} \frac{1-z^{-1}}{1+z^{-1}/7}\right)^\alpha$	Al-Alaoui rule
$s^\alpha \approx \left(\frac{1-z^{-1}}{T}\right)^\alpha$	Backward Euler rule
$s^\alpha \approx \left(\frac{2}{3T} \frac{1-z^{-1}}{1+z^{-1}/3}\right)^\alpha$	Implicit adams second rule

2) Indirect method

The second one uses a continuous time approximations such Charef's approximation, Oustaloup's approximation and Carlson's approximation to obtain an integer-order transfer function then discretizes this function by z transformation. This method is suitable for the discretization of fractional-order transfer functions.

In this paper, we focus on the indirect method of discretization using Oustaloup's recursive approximation. Thus, we follow the steps below [8]:

- Choose the frequency range  $[w_l, w_h]$  and  $N$
- Calculate the parameters  $K$  using (9),  $w_k$  using (8) and  $w'_k$  using (7).
- From (6), we obtain the rational approximation of  $s^\alpha$
- Replace the fractional operator  $s^\alpha$ , by it rational approximation in the fractional order transfer function.
- Discretize the obtained transfer function by replacing 's' by the bilinear transform or El-Alaoui transform.

III. MULTI-MODEL APPROACH FOR MODELLING

To simplify studying complex systems and "black box" systems, the decomposition of such system to a set of simple and linear models is a remedy. This decomposition approach is named multi-model approach.

Using multi-model approach for modeling allows a better understanding of the system behavior by local simple models; each model describes the system behavior in a limited functioning zone.

Since there is no information, we use classification methods, generally neuronal networks, to determine functioning zone.

Some researchers emphasize the use of KOHONEN Map in this step such as [14] and [16].

This method is known by the abbreviation SOM (Self-Organizing Map). The KOHONEN network is an automatic classification technique (unsupervised learning) that allows regrouping similar data in clusters taking into account that data belonging to different clusters are different. In [16], the authors use this technique to generate basis of models for an uncertain linear system. Besides, they prove its performances by experimental validation on an olive oil esterification reactor.

After determining the number of classes and choosing the order of local model according to [14], the next step is the identification of local models using recursive least square algorithm.

The validity coefficients are, then, computed, they allow to generate an output similar to the real output of the “black box” system. Subsequently, the capacity of local model to imitate the real system behavior in a functioning zone is determined according to [14] and [13]. Many techniques are developed to calculate validities [14], [17], [18] and [19]; the most useful is the residue approach.

#### IV. FRACTIONAL SYSTEMS MODELING USING MULTI-MODEL APPROACH

In this paper, we choose to study the following fractional order transfer function:

$$F(s) = \frac{b_0}{s^\alpha + a_0} \quad (11)$$

This transfer function is suitable for some physical systems [7] including beam heating process [9], thermal systems [10] and electrical networks [11]. This model was used as a benchmark system in [12]

In order to control the fractional order system represented by (11) using control law developed in [13], we need to present it using multi-model approach.

This representation reduces the complexity of this type of systems and allows the extensions of methods, conventionally used with linear systems, to fractional order systems. [13]

The First step is the discretization of the transfer function (11) using the indirect method described below. The obtained function is linear but of high order.

The second step is the use of KOHONEN Map Classification [14] to define the functioning zones of local models, then the use of recursive least squares [15] to identify local models.

The next step is the calculation of validities by residue approach [13].

Finally, the multi-model output is equal to the weighted sum of local outputs.

$$Y_{MM} = \sum_{i=1}^N V_i Y_i \quad (12)$$

Where  $n$  is the number of local models,  $V_i$  is the validity of the local model  $n^\circ i$  and  $Y_i$  is the output of the local model  $n^\circ i$ .

#### V. SIMULATION RESULTS AND DISCUSSIONS

This section includes the application of the theoretical steps described in the previous section to the following fractional order transfer function:

$$F(s) = \frac{0.09}{s^{0.5} + 0.3} \quad (13)$$

This model represents the identification result of the diffusive heat transfer phenomena of a brass ball with 3 cm radius, given in [10].

##### A. Discretization

Firstly, we apply the equations (6), (7), (8), (9) and (10) to the fractional operator ‘ $s^{0.5}$ ’ where  $N=3$ , the range of frequency is [0.01...1000], we obtain the following function:

$$s^{0.5} \approx \frac{56.23s^3 + 10^4 s^2 + 8293s + 31.62}{s^3 + 2622 s^2 + 3.177 \cdot 10^4 s + 1778} \quad (14)$$

After that, we substitute  $s^{0.5}$  by (14) in the equation (13).

The rational approximation is then given:

$$F(s) = \frac{0.09s^3 + 236s^2 + 2859s + 160}{56.53s^3 + 1.083 \cdot 10^4 s^2 + 1.782 \cdot 10^4 s + 565.1} \quad (15)$$

By applying the discretization using  $z$  transformation, we obtain the following function:

$$F(z^{-1}) = \frac{0.02965 - 0.007564z^{-1} - 0.02896z^{-2} + 0.007128z^{-3}}{1 - 1.037z^{-1} - 0.6478z^{-2} + 0.6855z^{-3}} \quad (16)$$

Where the sampling period  $T_s = 0.1$  s.

To obtain measurements of inputs/ outputs for identification, we apply to the transfer function given by (16) a pseudo-random signal rich in amplitude and frequency.

##### B. Multi-model approach

###### 1) Basis of models Determination

Our system is considered as a black box, so we use classification by KOHONEN Map to determine different functioning zones. This method was introduced and detailed in [14] where authors used KOHONEN Map to classify measurements. They obtained a well-distinguished grouping. Each grouping presents a local model. Then, they identified the local models using recursive least square.

In our simulation example, the Self-organization Map analysis allows a visual identification of the different classes which presents the number of local models. Classes are shown in figure 1:

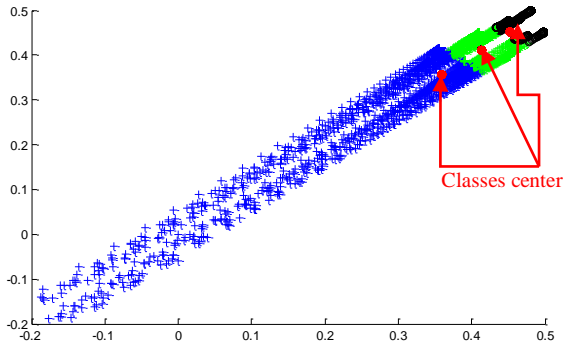


Fig1: Kohonen Map for measured data of the simulation example after learning

Figure 1 depicts the number of classes; each group presents a local model. So we can distinguish three classes and three local models.

After applying Recursive least square algorithm, we identify the following models:

$$F1(z^{-1}) = \frac{-0.00209z^{-1} - 0.01865z^{-2}}{1 - 0.8003z^{-1} - 0.2689z^{-2}} \quad (17)$$

$$F2(z^{-1}) = \frac{-0.002956z^{-1} + 0.01276z^{-2}}{1 - 1.841z^{-1} + 0.7707z^{-2}} \quad (18)$$

$$F3(z^{-1}) = \frac{0.02819z^{-1} - 0.02327z^{-2}}{1 + 0.1244z^{-1} - 1.089z^{-2}} \quad (19)$$

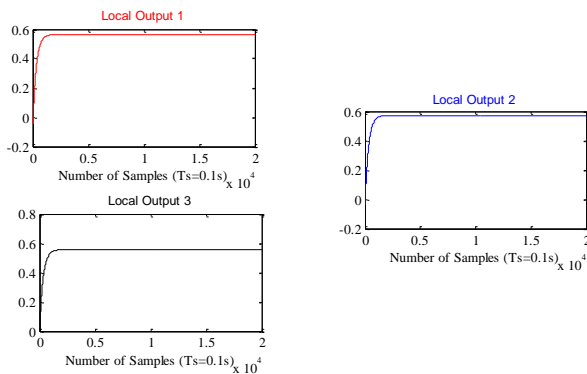


Fig 2: Local outputs

The figure 2 illustrates the step response of the obtained local models.

1) Calculus of validities by residue approach

The validities of each model is given by :

$$v_i = \frac{1 - r_{in}}{N - 1} \quad (20)$$

Where N present the number of local models,  $r_{in}$  is the normalized residue given by :

$$r_{in} = \frac{\|y - y_i\|}{\sum_{i=1}^N \|y - y_i\|} \quad (21)$$

To obtain the best result, we use the notion of strengthened validity. Enhanced validity is given by the following formula:[13]

$$v_{irenf} = v_i * \prod_{j=1, i \neq j}^N (1 - v_j) \quad (22)$$

And the normalized enhanced validity is calculated by the formula:

$$v_{i\_renfN} = \frac{v_{irenf}}{\sum_{i=1}^N v_{irenf}} \quad (23)$$

Figure 3 depicts the validity of each local model.

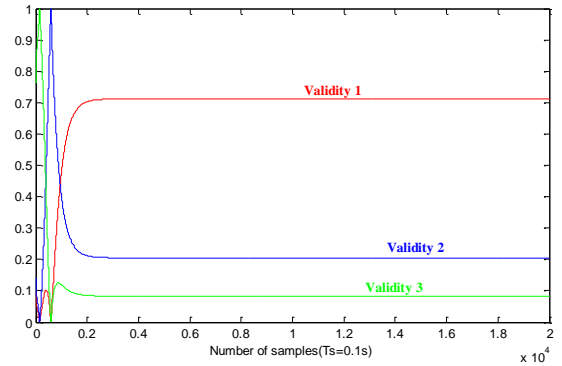


Fig 3: Validities of local models

2) Multi-model output

The multi-model approach is obtained by applying (12).

The obtained multi-model output coincides with the real output as shown in figure 4

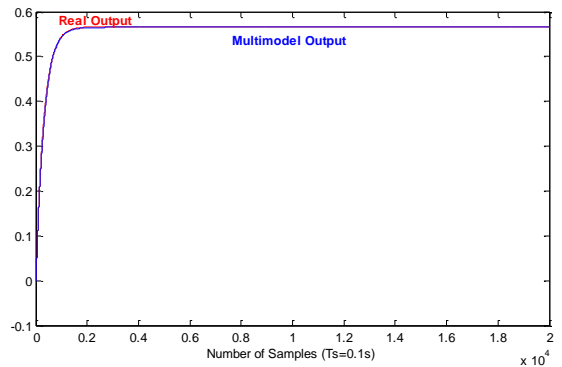


Fig.4: Evolution of Real output and multi-model output

C. Discussions

Using Self-organizing Map (KOHONEN Map) facilitates the determination of simple local models from only outputs measurements (black box system).

The effectiveness of our proposed method is proved as shown in figure 4: it presents a comparison between the multi-model output and the real output. The error between these two outputs is shown at figure 5

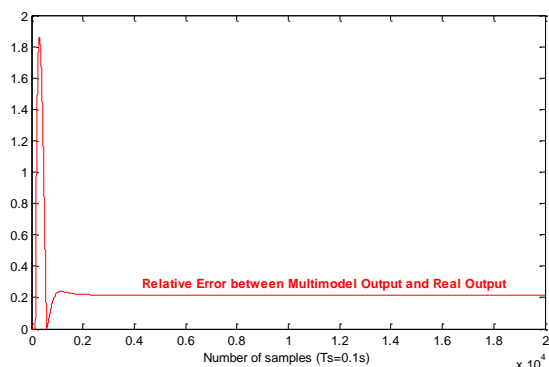


Fig5. Evolution of Relative Error between multi-model output and real output.

It is clear that the relative error is relatively small and the multi-model output represents with very high precision the real output. As a consequence, the effectiveness of our proposed method is confirmed.

#### V. CONCLUSIONS

This work emphasizes the advantages of multi model approach and the possibility to designing RST controller for fractional order system using multi-model approach [13].

In particular, a procedure to model fractional order system using a multi-model approach is presented. The determination of the model's basis is based on using the Self-organizing Map (KOHONEN Map) as a method for classification and the recursive least square algorithm for the identification of local models. The proposed method is applied to a fractional transfer function modelling a diffusive heat transfer of a brass ball with 3cm radius identified in [10]. The obtained multi-model output describes well the real output of the thermal system and the relative error is considerably small which proves the effectiveness of this method.

As a perspective, we suggest to use the proposed approach for the control of fractional order systems.

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